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Reduction Of Discrete Time Systems Using Routh Hurwitz Array And Error Minimization Technique

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Abstract: A new mixed method is presented for the reduction of discrete linear time invariant high order systems. This method combines the advantages of Routh Hurwitz array and error minimization technique. In the proposed method, the denominator of the reduced model is obtained by Routh Hurwitz array method and the numerator terms of the lower order transfer function are obtained by minimizing the Integral Square Error (ISE) between the step responses of the original system and the reduced model. The error function is converted into the frequency domain and the minima operation is carried out in this domain to minimize the error function. The proposed method is illustrated by a concrete example.

1 INTRODUCTION

Physical systems can be described mathematically by a large number of high order differential equations. These equations lead to a high order transfer function in the frequency domain or when decoupled into a system of first order differential equations, a high order state space model is obtained in time domain. The exact analysis of most of high order systems is both tedious and costly. Thus a high order system possesses a great challenge to both system analyst and control system designer and therefore, it is always desirable to replace such a high order system by a system of lower order that retains the important characteristics (stability, time response, frequency response) of the original system under consideration

A number of different methods [1-7] are available in the literature for the reduction of high order continuous time systems but only a very few [8-11] are available to reduce discrete time systems. In the recent paper [6], the author introduced a mixed method to obtain reduced order models for continuous time SISO (single input single output) systems using the advantages of Routh Hurwitz array method and step response error minimization technique. This method is extended [7] to reduce continuous time MIMO (multi input multi output) systems. The proposed paper is the extension of the work [6]. In the present paper, the method [6] is extended to reduce the discrete time systems.

In the proposed method, for retaining the stability of the system, the denominator of the reduced model is obtained by Routh-Hurwitz array method [6,12] and then the numerator terms of the reduced model are obtained so that the Integral Square Error (ISE) function between the step responses of the original system and the reduced model is minimum while the steady state equality condition is satisfied. The error function is converted into the frequency domain and the minima operation is carried out in this domain to minimize the error function. The proposed method is illustrated by an example and the unit step responses of original and reduced systems have been compared using MATLAB Software Package.

2 STATEMENT OF PROBLEM:

Let an n^{th} order linear time-invariant SISO discrete time system be described in frequency domain by z - transfer function

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$$G(z) = \frac{\sum_{i=0}^{n-1} a_i^* z^i}{\sum_{i=0}^{n} b_i^* z^i}$$
(1)

where $a_i^*(0 \le i \le n-1)$ and $b_i^*(0 \le i \le n)$ are scalar constants.

The problem is to find a $r^{th}\,(r < n)$ order reduced model $R_r(z)$ of the form :

$$R(z) = \frac{\sum_{i=0}^{r-1} c_i^* z^i}{\sum_{i=0}^{r} d_i^* z^i} \qquad (2)$$

where $c_i^*(0 \le i \le r-1)$ and $d_i^*(0 \le i \le r)$ are scalar constants, so that the reduced model retains the important characteristics of the original system and approximates its response as closely as possible for the same type of inputs.

The proposed model reduction method:

The Model Reduction Method consists of following steps :

Step 1 : Transformation of G(z) into w -domain :

Firstly transform G(z) into w-domain by using bilinear transformation $z = (1+w)/(1-w)$ as

$$H(w) = G(z)\Big|_{z=\frac{1+w}{1-w}} = \alpha + G^*(w) \qquad (3)$$

where

$$\alpha = H(w)\Big|_{w \to \infty} \qquad (4)$$

and

$$G^*(w) = \frac{N(w)}{D(w)} = \frac{\sum_{i=0}^{n-1} a_i w^i}{\sum_{i=0}^{n} b_i w^i} \qquad (5)$$

Here $a_i(0 \le i \le n-1)$ & $b_i(0 \le i \le n)$ are scalar constants.

Step 2 : Order reduction of $G^*(w)$ in w - domain

Now the system $G^*(w)$ is reduced to an r^{th} order model $R^*(w) = \frac{N_r^*(w)}{D_r(w)} = \frac{\sum_{i=0}^{r-1} c_i w^i}{\sum_{i=0}^{r} d_i w^i}$ in w - domain using the method [6,12] in the following way –

1) **Determination of Denominator $D_r(w)$ using Routh Hurwitz Array Method**

Using Routh-Hurwitz array [6,12], we obtain the denominator of the reduced model by constructing denominator stability array

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Denominator Stability Array

$b_{1,1} = b_n$	$b_{1,2} = b_{n-2}$	$b_{1,3} = b_{n-4}$	$b_{1,4} = b_{n-6}$
$b_{2,1} = b_{n-1}$	$b_{2,2} = b_{n-3}$	$b_{2,3} = b_{n-5}$	$b_{2,4} = b_{n-7}$
$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	…….
$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	…….
$b_{5,1}$	$b_{5,2}$	…….	
$b_{6,1}$	$b_{6,2}$	…….	
…….	…….		
…….	…….		
$b_{n,1}$			
$b_{n+1,1}$			

The above array is completed in the conventional way by computing the coefficients of succeeding rows by the algorithm

$$b_{i,j} = b_{i-2,j+1} - \frac{b_{i-2,j} b_{i-1,j+1}}{b_{i-1,1}} \qquad \begin{array}{l} 3 \le i \le n+1 \\ 1 \le j \le [(n-i+3)/2], \end{array}$$

(6)

where [x] stands for the integral part of the quantity x.

The r^{th} order reduced denominator $D_r(w)$ is then given by

$$D_r(w) = b_{n+1-r,1} w^r + b_{n+2-r,1} w^{r-1} + b_{n+1-r,2} w^{r-2} + b_{n+2-r,2} w^{r-3} + \dots\dots\dots\dots + k \qquad (7)$$

where

$$k = \begin{cases} b_{n+1-r,\, r/2+1} & if\ r\, is\, even \\ b_{n+2-r,\, (r-1)/2+1} & if\ r\, is\, odd \end{cases}$$

After normalizing (7), the obtained reduced denominator $D_r(w)$ may be rewritten in the form $D_r(w) = \sum_{i=0}^{r} d_i w^i$

(8)

2) Determination of Numerator $N_r^*(w)$ using Error Minimization Technique

In terms of the poles, $G^*(w)$ and $R^*(w)$ may be rewritten as

$$G^*(w) = \frac{k_0'}{w+p_1} + \frac{k_1'}{w+p_2} + \dots\dots\dots\dots\dots + \frac{k_{n-1}'}{w+p_n} = \sum_{u=0}^{n-1} \frac{k_u'}{w+p_{u+1}}$$

(9)

$$R^*(w) = \frac{l_0'}{w+q_1} + \frac{l_1'}{w+q_2} + \dots\dots\dots\dots\dots + \frac{l_{r-1}'}{w+q_r} = \sum_{u=0}^{r-1} \frac{l_u'}{w+q_{u+1}}$$

(10)

where we have assumed that the original system has non repeating distinct simple poles $-p_1, -p_2, \dots\dots\dots\dots, -p_n$ on the real axis in the left half of complex plane while the reduced model has $-q_1, -q_2, \dots\dots\dots\dots, -q_r$ as its distinct simple poles in the complex plane.

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In order to determine the numerator $N_r^*(w)$ of the reduced model $R^*(w)$, the step response error function, for the step responses $y(t)$ and $y_r(t)$ of the system $G^*(w)$ and that of reduced model $R^*(w)$, is

$E = \int_0^\infty [y(t) - y_r(t)]^2 dt$, which may be expressed as

$$E = \int_0^\infty [h(t) - h_r(t)]^2 dt$$
(11)

where, $h(t) = y(\infty) - y(t)$ and $h_r(t) = y_r(\infty) - y_r(t)$ are the transient parts of the step responses of the system $G^*(w)$ and the reduced model $R^*(w)$ respectively.

The matching condition of steady state values of the reduced model $R^*(w)$ and the system $G^*(w)$, i.e., $y_r(\infty) = y(\infty)$ leads to

$$\frac{k_0^{'}}{p_1} + \frac{k_1^{'}}{p_2} + \dots\dots\dots\dots + \frac{k_{n-1}^{'}}{p_n} = \frac{l_0^{'}}{q_1} + \frac{l_1^{'}}{q_2} + \dots\dots\dots\dots + \frac{l_{r-1}^{'}}{q_r}$$
(12)

Now,

$$H_o(w) = L\{h(t)\} = \frac{y(\infty)}{w} - \frac{G^*(w)}{w} = \frac{k_0}{w + p_1} + \frac{k_1}{w + p_2} + \dots\dots\dots\dots + \frac{k_{n-1}}{w + p_n}$$
$$= \sum_{u=0}^{n-1} \frac{k_u}{w + p_{u+1}} \quad \text{where } k_u = \frac{k_u^{'}}{p_{u+1}}$$
(13)

Similarly, in view of equations (10) and (12), we have

$$H_r(w) = L\{h_r(t)\} = \frac{y_r(\infty)}{w} - \frac{R^*(w)}{w} = \frac{l_0}{w + q_1} + \frac{l_1}{w + q_2} + \dots\dots\dots\dots + \frac{l_{r-1}}{w + q_r}$$
$$= \sum_{u=0}^{r-1} \frac{l_u}{w + q_{u+1}} \quad \text{where } l_u = \frac{l_u^{'}}{q_{u+1}}$$
(14)

Using (13) and (14) in (12), we get

$$k_0 + k_1 + \dots\dots\dots\dots + k_{n-1} = l_0 + l_1 + \dots\dots\dots\dots + l_{r-1}$$
(15)

Now using Parseval's theorem [13], we see that the step response error function E can be expressed as

$$E = I_1 + I_2 - 2I_3$$
(16)

where, $I_1 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} H_o(w) H_o(-w) dw$,
(16a)

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$$I_2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_r(w)H_r(-w)\,dw,$$

(16b)

and $$I_3 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} H_o(w)H_r(-w)\,dw.$$

(16c)

Following Newton et al. [13] the integrals I_1, I_2 and I_3 may be evaluated and be put in summation form as

$$I_1 = \sum_{u=0}^{n-1}\sum_{v=0}^{n-1}\frac{k_u k_v}{p_{u+1}+p_{v+1}} = \text{constant} = k \text{ (say)}, \tag{17}$$

$$I_2 = \sum_{u=0}^{r-1}\sum_{v=0}^{r-1}\frac{l_u l_v}{q_{u+1}+q_{v+1}}, \tag{18}$$

and $$I_3 = \sum_{u=0}^{n-1}\sum_{v=0}^{r-1}\frac{k_u l_v}{p_{u+1}+q_{v+1}}. \tag{19}$$

It is to be noted here that k_u, p_u & p_v are known constants that is why I_1 equals to some constant k.

Thus in view of equations ((16)-(19)), the step response error function can be expressed as a function of the variables $l_0, l_1, \ldots\ldots\ldots, l_{r-1}$ in the form

$$E = k + \sum_{u=0}^{r-1}\sum_{v=0}^{r-1}\frac{l_u l_v}{q_{u+1}+q_{v+1}} - 2\sum_{u=0}^{n-1}\sum_{v=0}^{r-1}\frac{k_u l_v}{p_{u+1}+q_{v+1}} \tag{20}$$

where k_u, p_u, p_v, q_u & q_v are already known constants. Now the minimization conditions for the step response error function E, namely,

$$\frac{\partial E}{\partial l_u} = 0 \qquad u = 0,1,2,\ldots\ldots\ldots\ldots, r-2. \tag{21}$$

a set of $(r-1)$ linear equations

$$\begin{aligned}
&\frac{l_0}{q_1} + 2\frac{l_1}{q_1+q_2} + \ldots\ldots\ldots + 2\frac{l_{r-2}}{q_1+q_{r-1}} + 2\frac{l_{r-1}}{q_1+q_r} - 2\left(\frac{k_0}{p_1+q_1} + \frac{k_1}{p_2+q_1} + \ldots\ldots\ldots\ldots + \frac{k_{n-1}}{p_n+q_1}\right) = 0\\
&2\frac{l_0}{q_2+q_1} + \frac{l_1}{q_2} + \ldots\ldots\ldots + 2\frac{l_{r-2}}{q_2+q_{r-1}} + 2\frac{l_{r-1}}{q_2+q_r} - 2\left(\frac{k_0}{p_1+q_2} + \frac{k_1}{p_2+q_2} + \ldots\ldots\ldots + \frac{k_{n-1}}{p_n+q_2}\right) = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&2\frac{l_0}{q_{r-1}+q_1} + 2\frac{l_1}{q_{r-1}+q_2} + \ldots + \frac{l_{r-2}}{q_{r-1}} + 2\frac{l_{r-1}}{q_{r-1}+q_r} - 2\left(\frac{k_0}{p_1+q_{r-1}} + \frac{k_1}{p_2+q_{r-1}} + \ldots + \frac{k_{n-1}}{p_n+q_{r-1}}\right) = 0
\end{aligned} \tag{22}$$

in $l_0, l_1, l_2, \ldots\ldots\ldots, l_{r-1}$.

Now equations (22) together with (15) give the values of $l_0, l_1, l_2, \ldots\ldots\ldots, l_{r-1}$, which in view of (14) yield the unknown constants $l_0^{'}, l_1^{'}, l_2^{'}, \ldots\ldots\ldots, l_{r-1}^{'}$. Thus, we get the r^{th} order reduced model $R^*(w)$ as given by (10).

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Consequently the r^{th} order model $H^*(w)$ in w - domain is given by

$$H^*(w) = \alpha + R^*(w) = \alpha + \frac{N_r^*(w)}{D_r(w)} \tag{23}$$

Step 3 : Retransformation of $H^*(w)$ into z - domain

Now $H^*(w)$ is retransformed back into z - domain by using inverse bilinear transformation, $w = (z-1)/(z+1)$ to yield

$$R^\dagger(z) = H^*(w)\Big|_{w=\frac{z-1}{z+1}} = \beta + R^\#(z) \tag{24}$$

where

$$\beta = R^\dagger(z)\Big|_{z\to\infty} \tag{25}$$

and

$$R^\#(z) = \frac{N_r^\#(z)}{D_r(z)} \tag{26}$$

To match the rank of original and reduced systems, the constant β is neglected. It is done because due to the nature of the bilinear transformation, the initial value of the step response of the reduced order model may not be zero even though the initial value of the step response of the original system is zero.

Step 4 : Matching of steady state values

In the process of transforming from z - domain to w - domain and then back from w - domain to z - plane, there arises steady state error between the original system $G(z)$ and its r^{th} order reduced model $R^\#(z)$. To remove this error, the gain correction factor K is calculated as

$$K = \frac{SSO}{SSR} = \frac{G(z)}{R^\#(z)}\Bigg|_{z=1} \tag{27}$$

where SSO and SSR are the steady state values of $G(z)$ and $R^\#(z)$ respectively.

Now multiplying $N_r^\#(z)$ by K, the numerator $N_r(z)$ of the reduced order model $R(z)$ is obtained and thus the r^{th} order reduced model $R(z) = N_r(z)/D_r(z)$ is obtained in the form of (2).

Example:

As an illustration to the above method Consider the fourth order discrete time system [14], whose z - transfer function is given by

$$G(z) = \frac{2z^3 - 4.424z^2 + 3.3634z - 0.8721}{z^4 - 3.037z^3 + 3.425z^2 - 1.693524z + 0.308433}$$

(28)

Using the proposed method, the second and third order reduced models are given by

$$R_2(z) = \frac{1.395191z + 1.105077}{z^2 - 1.855917z + 0.868457}$$

and

$$R_3(z) = \frac{1.565333z^2 - 2.577014z + 1.095756}{z^3 - 2.677951z^2 + 2.410938z - 0.729353}$$

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The unit step responses of the original system and the reduced models drawn by MATLAB software package are shown in Fig.1.

3 CONCLUSIONS:

A new model order reduction method has been presented in this paper to approximate a high order z transfer function $G(z)$ by a low order one $R(z)$. This method is computationally simple and efficient. It gives the stable reduced system if the original one is stable. The method is a generalized one and can be used to obtain a reduced model of any order r, which may be even or odd, for a given system of any order n. This method has also been successfully tried to reduce the multivariable discrete time systems.

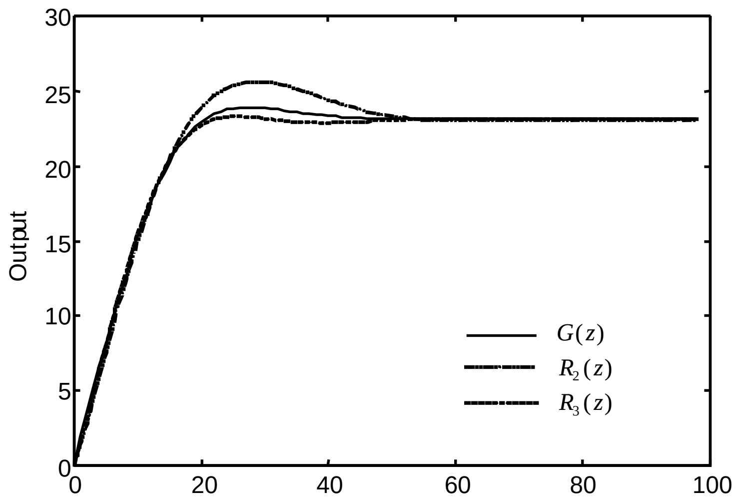

Fig. 1: Comparison of unit step responses

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