

The T-X Family of Distributions: A Retrospect

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Abstract-The T-X method for generating new distributions is an emerging area in recent statistical literature. As a result, new distributions have been developed and studied. This article is a continuation of the paper "A Review Study on T-X Family of Distributions" by Tomy and Jose (2018). It reviews some special members of T-X family of distributions such as, Kumaraswamy-Geometric distribution, T-Normal Family, Poisson-X family, T-Transmuted family and Kumaraswamy-Marshall Olkin family of distributions.

Keywords: Transmuted family, Marshall-Olkin distribution, T-X family, Truncated Poisson distribution, Method of Maximum Likelihood

1. INTRODUCTION

In the present scenario of big data analytics, statistical modeling has got paramount importance. Researchers have started exploring innovative paths by introducing new distributions so as to fit the data in hand.

Alzaatreh et al.[1] proposed a general method to generate new family of distributions called "Transformed-Transformer" method and the family is called T-X family of distributions. The generator T can be any continuous distribution. During the milestones, the Beta generated method laid the first stone. In similar vein Kumaraswamy generated, Gamma generated, Kummer beta generalized and Generalized beta generated distributions paved the further path. The support of the generator in Beta-X and Kumaraswamy generated(Kw-G) family is 0 to 1, the transformed-transformer method overcomes this limitation and use any continuous distribution as a generator to generate distributions by using the function $W[F(x)]$. Many subfamilies of T-X family have been studied in the literature, some of them are Burr X family, T-Geometric family, T-Normal family, Half-Logistic generated, Poisson-X family, T-Transmuted family, Beta Marshall-Olkin and Kumaraswamy Marshall-Olkin family of Distributions. This article is a continuation of the paper entitled "A Review Study on T-X Family of Distributions" by Tomy and Jose[2] discusses the distributions which paved the way for T-X family of distributions and some members of T-X family of distribution.

The main aim of this review is to study and discriminate different subfamilies of T-X family of distributions and provide available literature on continuous and discrete families.

The paper unfolds as follows: In section 2, we consider T-X Family of distributions. Section 3 discusses some special members of T-X family of distributions. Finally in Section 4 we conclude paper.

2. THE T-X FAMILY OF DISTRIBUTIONS

A large number of distributions can be generated by applying any two existing univariate distributions based on T-X method. Several known continuous and discrete distributions are found to be special cases of the first generated distributions. Let $r(t)$ be the Probability Density Function(PDF) and $R(t)$ be the Cumulative Density Function(CDF) of a random variable $T \in [a, b]$ for $-\infty < a < b < \infty$ and let $W[F(x)]$ be a function of the CDF

$F(x)$ of some baseline continuous random variable X so that $W[F(x)]$ satisfies the following conditions:

$$W[F(x)] \in [a, b]$$

$W[F(x)]$ is differentiable and monotonically non decreasing

$$W[F(x)] \rightarrow a \text{ as } x \rightarrow -\infty \text{ and } W[F(x)] \rightarrow b \text{ as } x \rightarrow \infty$$

The CDF of a new family of distributions is defined as

$$G(x) = \int_a^{W[F(x)]} r(t) dt \quad (2.1)$$

where $W[F(x)]$ satisfies the above conditions. The CDF $G(x)$ in equation(2.1) can be written as

$G(x) = R\{W[F(x)]\}$, where $R(t)$ is the CDF of the random variable T . If X is a continuous random variable the corresponding PDF is given by

$$g(x) = \left\{ \frac{d}{dx} W[F(x)] \right\} r\{W[F(x)]\} \quad (2.2)$$

If X is a discrete random variable, the T - X family, is a family of discrete distributions. The probability mass function (PMF) of the T - X family of discrete distributions may now be written as

$$g(x) = G(x) - G(x-1) = R\{W[F(x)]\} - R\{W[F(x-1)]\}. \quad (2.3)$$

The term "Transformed-Transformer" was used since the PDF $r(t)$ in equation(2.1) is "Transformed" into a new PDF $g(x)$ through the function $W[F(x)]$, which act as a "transformer", that is, the random variable X is used to transform another random variable T . Different $W(F(x))$ will define based on the support of the random variable T .

Table 1 lists the T - X families for different $W[F(x)]$ function.

Table 1: T - X families based on different $W[F(x)]$

$W[F(x)]$	Range of T	Members of T - X family
$F(x)$	$[0, 1]$	Beta Generated (Eugene et al.[3])
		Kw-G (Cordeiro and de-Castro.[4])
		Kummer Beta Generated(KBG) (Pescim et al.[5])
		McDonald Generated (Alexander et al.[6])
		Poisson-X (Tahir et al.[7])
		Kumaraswamy-geometric (Akinsete et al.[8])
$\frac{F(x)}{c + (1-c)F(x)}$	$[0,1]$	Beta Marshall Olkin (Alizadeh et al.[9])
		Kumaraswamy Marshall Olkin (Alizadeh et al.[10])
$-\log [F(x)]$	$(0, \infty)$	Gamma generated (Ristic and Balakrishnan[11])
$-\log [1 - F(x)]$	$(0, \infty)$	Gamma Generated(Zografos and Balakrishnan[12])
		Weibull-X (Alzaatreh et al[1])
		Weibull-Pareto(Alzaatreh et al.[13])
		Lomax-G (Cordeiro et al.[14])
		Exponentiated Half-Logistic Generated (EHL-G) (Cordeiro et al.[15])
		Generalized Burr X family of distributions (Jamal and Nasir[16])
		Weibull-Lomax (Alzaghal et al.[17])
		ExponentiatedGompertz Generated (Cordeiro et al.[18])
		Gamma-Pareto(Alzaatreh et al.[19])
		Exponentiated Exponential-Geometric (Alzaatreh et al.[20])

		Weibull-Power Cauchy (Tahir et al.[21])
$-\log [1 - F^\alpha(x)]$	$(0, \infty)$	Exponentiated T-X (Alzaghal et al.[22])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Gamma Generated (Torabi and Hedesh[23])
		A New Weibull-Pareto (Tahir et al.[24])
$-\log \left[1 - \frac{F(x)}{c + (1 - c)F(x)} \right]$	$(0, \infty)$	T-Transmuted X Family (Jayakumar and Girish[25])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Weibull Rayleigh(Merovci and Elbatal[26])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Burr X-G (BX-G) family of distributions (Yousof et al.[27])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Weibull-G (Bourguignon et al.[28])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Odd Log-Logistic-G(Cordeiro et al.[29])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Weibull-Exponential (Oguntunde et al.[30])
$\frac{F(x)}{1 - F(x)}$	$(0, \infty)$	Weibull-Inverse Exponential (Chandrakant et al.[31])
$\frac{F^\beta(x)}{1 - F^\beta(x)}$	$(0, \infty)$	Generalized Odd Gamma-G Family (Hosseini et al.[32])
$\log\{\frac{F(x)}{1 - F(x)}\}$	$(-\infty, \infty)$	Logistic-G (Torabi and Montazeri,[33])
$\log\{\frac{F(x)}{1 - F(x)}\}$	$(-\infty, \infty)$	Gumbel-Weibull(Al-Aqtash et al.[34])
$\log\{-\log[1 - F(x)]\}$	$(-\infty, \infty)$	Logistic-X (Tahir et al.[35])

3. SPECIAL SUBFAMILIES OF T-X FAMILY

In this section, we discuss some special cases of T-X family of distributions and their literatures are provided, such as T-Normal Family of distributions, Poisson-X family of distributions, Kumaraswamy-Geometric distribution, T-Transmuted family and Kumaraswamy-Marshall Olkin family of distributions.

3.1 T-NORMAL FAMILY OF DISTRIBUTIONS

Aljarrah et al.[36] used $W[F(x)]$ to be the quantile function of a random variable Y in T-X family and defined the T-X{Y} family for generating families of continuous probability distributions, the corresponding CDF is

$$G(x) = \int_a^{Q_Y[F(x)]} r(t)dt = R[Q_Y(F(x))] \quad (3.1)$$

where $Q_Y(p)$ is the quantile function of the random variable Y, $0 < p < 1$.

$$Q_Y(p) = \inf\{Y \in R : F(y) \geq p\}, \quad p \in (0,1) \quad (3.2)$$

If the function F is continuous and strictly monotonically increasing then $Q_Y = F^{-1}$. The above family is a family of continuous probability distributions then the PDF associated with equation(3.1) is

$$g(x) = \frac{f(x)}{f_Y\{Q_Y[F(x)]\}} r\{Q_Y[F(x)]\} \quad (3.3)$$

The quantile function of the random variable X with PDF g(x) is

$$Q_X(p) = Q_X(F_Y(Q_T(p))) \quad (3.4)$$

Where $Q_X(F_Y(Q_T(p)))$ is quantile function of the random variable X with PDF f(x) with

$$p = F_Y(Q_T(p)).$$

Some remarks are

(a) If the random variables X and Y with PDF $f(x)$ and $f(y)$ are identically distributed then

$$G = R.$$

(b) If the random variables T and Y are identically distributed then $G = F$.

(c) Beta-X, Kw-G and Generalized Beta Generated Distributions are belong to T-X{uniform} family.

One of the main advantage of using the T-X{Y} framework is that one can keep one or more parameters from the distribution of Y . If the parameter is shape parameter which increase the flexibility of new distribution.

Alzaatreh et al.[37] define the families of Generalized Normal(GN) distributions called T-Normal family which is a subfamily of T-X{Y} family of distribution. That is, X is a normal random variable with PDF $f(x) = \phi(x)$ and CDF $F(x) = \Phi(x)$, then the CDF of T-normal{Y} family of distributions as

$$G(x) = \int_a^{Q_Y[\Phi(x)]} r(t)dt = R[Q_Y(\Phi(x))] \quad (3.5)$$

The corresponding PDF is

$$g(x) = \frac{\phi(x)}{f_Y\{Q_Y[\Phi(x)]\}} r\{Q_Y[\Phi(x)]\} \quad (3.6)$$

Some special cases are

(a) The beta normal, Kumaraswamy normal and generalized beta-generated normal are belong to the T-normal{standard uniform} family.

(b) The gamma-normal distribution studied by Alzaatreh et al.[37] is a member of T-normal{standard exponential} family.

The Hazard Rate Function (HRF) of the T-normal{Y} family is given by

$$h(x) = \phi(x) \frac{r(Q_Y(\Phi(x)))}{f_Y(Q_Y(\Phi(x)))} \quad \text{where} \quad \phi(x) = \frac{\phi(x)}{1-\Phi(x)}$$

Some examples of GN families of distributions with different choices for the random variables Y and T are Weibull-N{exponential}, logistic-N{log-logistic}, exponential-N{log-logistic}, exponentiated-exponential-N{uniform} and logistic-N{extreme value}.

3.1.1 WEIBULL-N{EXPONENTIAL} DISTRIBUTION

In this section, we define and study some properties of a member of the T-Normal family, the Weibull-N{Exponential} distribution introduced by Alzaatreh et al.[37]

If a random variable T follows the Weibull distribution with parameters c and β with PDF

$$r(t; c, \beta) = (c/\beta)(t/\beta)^{(c-1)}e^{-(t/\beta)^c}, t \geq 0, c > 0, \beta > 0$$

From equation(3.5), the CDF of Weibull-N{exponential} is defined as

$$G(x) = 1 - e^{\{-[-(1/\beta)\log(1-\Phi(x))]^c\}} \quad (3.7)$$

By using equation(3.6), the PDF of the Weibull-N{exponential} is given by

$$g(x; c, \beta) = \frac{c}{\beta} \frac{\phi(x)}{1-\Phi(x)} \left\{ \frac{-\log(1-\Phi(x))}{\beta} \right\}^{c-1} e^{-\left\{ \frac{-\log(1-\Phi(x))}{\beta} \right\}^c}, x \in R, c > 0, \beta > 0 \quad (3.8)$$

Figure 1 gives the graph of PDF of Weibull-N{Exponential} distribution for various values of β and c .

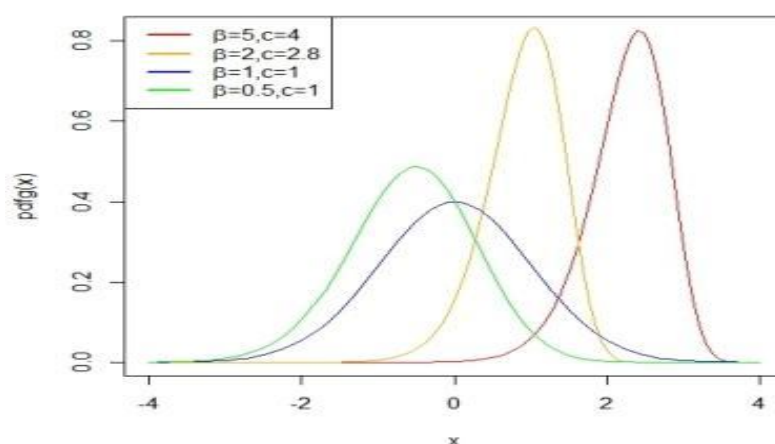


Figure 1: PDF of Weibull-N{exponential} for Various Values of β and c .

Some properties are

- (a) When $c = 1$, the Weibull-N{Exponential} reduces to the Exponential-Normal distribution with $\theta = \frac{1}{\beta}$.
- (b) When $c = \beta = 1$, the Weibull-N{Exponential} reduces to the Normal distribution.
- (c) It is left skewed whenever $\beta > 1$ and right skewed whenever $\beta < 1$. For fixed β , the peak increases as c increases.

The quantile function of the Weibull-N{exponential} distribution is given by

$$Q_X(p) = \Phi^{-1}\{1 - e^{-\beta(-\log(1-p))^{1/c}}\}$$

The shapes of GN distributions can be symmetric, skewed to the right, skewed to the left. This gives the families some flexibility in fitting real world data.

In literature, Nasir et al.[38] introduced and studied the properties of the T-Burr family of distribution as a special case of the T-X{Y} family of distribution. Alzaatreh et al.[39] introduced T-Gamma family has been proposed using the T -X{Y} framework. Similarly Almeida et al.[40] and Alzaatreh.[41] proposed T-Weibull{Y} family and T-Cauchy{Y} family.

3.2 THE POISSON-X FAMILY OF DISTRIBUTIONS

Tahir et al.[7] suggested the Poisson-X family of distributions which is a subfamily of T-X family of distribution but here T is not a Poisson random variable. The procedures used for obtaining the distribution of T are, Consider a company with N independent working systems and that each system consist of c parallel units. The random variables $Z_{i1}, Z_{i2}, \dots, Z_{ic}$, $i = 1, 2, \dots, N$ are the failure times of the parallel components of the i^t system, are independent and identically distributed standard uniform random variables and Y_i denote the failure time of the i^t system. Also N is a random variable having a zero truncated Poisson distribution with PMF

$$P(N = n) = \frac{\lambda^n}{n!(e^\lambda - 1)}, n = 1, 2, 3, \dots, \lambda > 0 \quad (3.9)$$

The random variable T represents the time to failure of the first out of the N functioning systems ie, $T = \min(Y_1, \dots, Y_N)$. Based on the these situations the obtained conditional CDF of T is

$$R(t|N) = 1 - (1 - t^c)^n \quad (3.10)$$

Then the CDF of T is,

$$\begin{aligned} R(t) &= \sum_{n=1}^{\infty} P(N=n)R(t|N) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!(e^\lambda-1)} 1 - (1-t^c)^n = \frac{1-e^{-\lambda t^c}}{1-e^{-\lambda}} \end{aligned} \quad (3.11)$$

The corresponding PDF is given by

$$r(t; \lambda, c) = \frac{\lambda c}{1-e^{-\lambda}} t^{c-1} e^{-\lambda t^c}; 0 \leq t \leq 1, \lambda > 0, c > 0 \quad (3.12)$$

if $\lambda = 1$ then the $r(t; \lambda, c)$ becomes

$$r(t; c) = \frac{c}{1-e^{-1}} t^{c-1} e^{-t^c}; 0 \leq t \leq 1, c > 0 \quad (3.13)$$

The CDF of the Poisson-X family is

$$G(x) = \int_0^{F(x)} r(t; c) dt = (1-e^{-1})^{-1} (1-e^{-[F(x)]^c}) \quad (3.14)$$

The corresponding PDF is

$$g(x; c) = \frac{c}{1-e^{-1}} f(x) F(x)^{c-1} e^{-[F(x)]^c}; c > 0 \quad (3.15)$$

3.2.1 THE POISSON POWER CAUCHY DISTRIBUTION

In this section, a member of the Poisson-X family namely the Poisson Power Cauchy Distribution (PPC) is considered and some of its structural properties are investigated.

Rooks et al.[42] proposed a two-parameter Power-Cauchy (PC) distribution, a submodel of the transformed beta distribution. The CDF and PDF of the PC distribution with shape parameter α and scale parameter σ is given by

$$F(x) = 2\pi^{-1} \tan^{-1}(x/\sigma)^\alpha \quad (3.16)$$

$$f(x; \alpha, \sigma) = 2\pi^{-1} (\alpha/\sigma) (x/\sigma)^{\alpha-1} [1 + (x/\sigma)^{2\alpha}]^{-1}; x > 0, \alpha > 0, \sigma > 0 \quad (3.17)$$

from equation(3.14) the CDF of PPC is defined as

$$G(x) = \frac{1}{(1-e^{-1})} (1 - e^{-[2\pi^{-1} \tan^{-1}(x/\sigma)^\alpha]^c}) \quad (3.18)$$

By using equation(3.15), the PDF of the PPC is given by

$$g(x; c, \alpha, \sigma) = \frac{2c\alpha(x/\sigma)^{\alpha-1} [2\pi^{-1} \tan^{-1}(x/\sigma)^\alpha]^{c-1}}{\sigma\pi(1-e^{-1}) [1 + (x/\sigma)^{2\alpha}]} \{e^{-[2\pi^{-1} \tan^{-1}(x/\sigma)^\alpha]^c}\}; x > 0, \alpha, \sigma, c > 0 \quad (3.19)$$

Figure 2 gives the graph of PDF of PPC distribution for various values of α , σ and c .

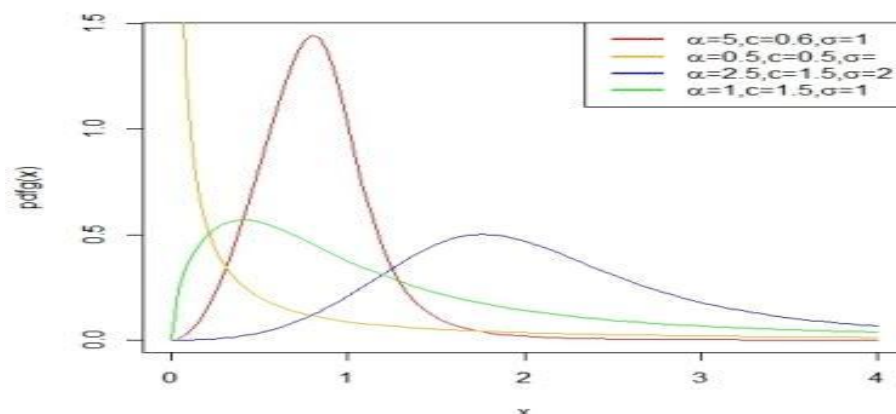


Figure 2: PDF of PPC Distribution for Various Values of α , σ and c .

The PPC distribution is unimodal and its density function can be expressed as a linear representation of Exponentiated Power Cauchy (EPC) densities. The PPC density can produce various shapes such as left-skewed, right-skewed and reversed-J.

The HRF $h(x)$ is

$$h(x) = \frac{2c\pi^{-1}(\alpha/\sigma)(x/\sigma)^{\alpha-1}[1+(x/\sigma)^{2\alpha}]^{-1}[2\pi^{-1}\tan^{-1}(x/\sigma)^{\alpha}]^c}{1-e^{-\{1-[2\pi^{-1}\tan^{-1}(x/\sigma)^{\alpha}]^c\}}} \quad (3.20)$$

It has increasing failure rate (IFR), decreasing failure rate (DFR), UBT (upside-down-bathtub) and BT (bathtub) shapes.

Tahir et al.[7] discovered the mathematical properties of this family such as ordinary and incomplete moments, quantile and generating functions, modes, Bonferroni and Lorenz curves of Poisson-X family. Also discussed the estimation of the parameters using maximum likelihood method. The distribution is applied to three real data sets were fitted for the PPC and compared with four known distributions. The results showed that the PPC is a relatively better model to fit data than the other distributions.

3.3 THE KUMARASWAMY-GEOMETRIC DISTRIBUTION

Akinsete et al.[8] proposed the Kumaraswamy-Geometric Distribution(KGD). It is obtained by taking T to be Kumaraswamy Distribution with CDF

$$R(t) = 1 - (1 - t^{\alpha})^{\beta} \quad 0 < t < 1, \alpha, \beta > 0$$

The transformer random variable X to have the geometric distribution and $W[F(x)] = F(x)$.

The CDF of KGD is

$$G(x) = \int_0^{F(x)} r(t)dt = 1 - [1 - (1 - p^{x+1})^{\alpha}]^{\beta} \quad (3.21)$$

The corresponding PMF for the KGD

$$g(x; p, \alpha, \beta) = [1 - (1 - p^x)^{\alpha}]^{\beta} - [1 - (1 - p^{x+1})^{\alpha}]^{\beta}; x = 0, 1, 2, \dots, 0 < p < 1, \alpha, \beta > 0 \quad (3.22)$$

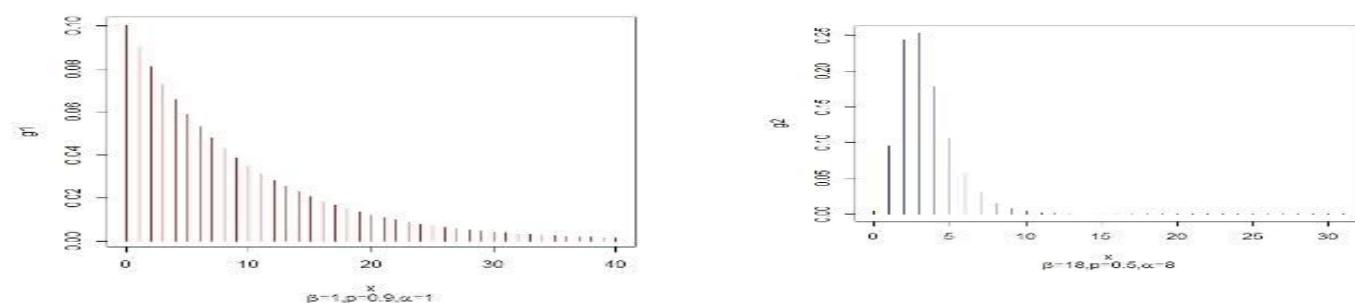


Figure 3 gives the graph of PMF of KGD distribution for various values of α, β and p .



Figure 3: PMF of KGD for Various Values of α, β and p .

When $\alpha = \beta = 1$, it gives the geometric distribution, when $\alpha = 1$, it gives the

geometric distribution with parameter $1 - p^\beta$ and when $\beta = 1$, it gives EEGD. Also KGD has a reversed J-shape with a unique mode. The KGD can also be derived by considering log-Kumaraswamy distribution instead of Kumaraswamy distribution and taking $W[F(x)] = -\log(1 - F(x))$.

The associated HRF is

$$(x) = \left\{ \frac{1 - (1 - p^x)^\alpha}{1 - (1 - p^{x+1})^\alpha} \right\}^\beta - 1 \quad (3.23)$$

The HRF of KGD, is increasing for $\alpha > 1$, decreasing for $\alpha < 1$ and at $\alpha = 1$ the HRF of KGD is constant.

Akinsete et al.[8] studied the properties of this family such as moments, quantile and generating functions and Shannon entropy are obtained. Also discussed the estimation of the parameters using maximum likelihood method. The distribution is applied to model two real life data sets were fitted for the KGD and compared with generalized negative binomial distribution (GNBD) defined by Jain and Consul [43] and Exponentiated Exponential-Geometric Distribution (EEGD). The results showed that the KGD is a relatively better model to fit data than the other distributions. Hamedani[44] studied the characterizations of KGD such as conditional expectation of certain function of the random variable and the reverse hazard rate function.

3.4 THE T-TRANSMUTED X FAMILY OF DISTRIBUTIONS

The Quadratic Transmuted Family of Distributions was proposed by Shaw and Buckley[45] with CDF

$$F(x) = (1 + \lambda)(J(x) - \lambda J^2(x))$$

for $\lambda \in [-1, 1]$, where $J(x)$ is the CDF of the baseline distribution.

Jayakumar and Girish[25] used $W[F(x)]$ to be the $-\log(1 - F(x))$ where $F(x)$ is the CDF of Quadratic Transmuted Family of Distributions, in T-X family of Distributions and declined the T-Transmuted X Family of Distributions for generating families of continuous probability distributions,

The corresponding CDF is

$$\begin{aligned} G(x) &= \int_0^{-\ln(1-F(x))} r(t) dt = \int_0^{-\ln(1-J(x)(1+\lambda J(x)))} r(t) dt \\ &= R\{-\ln(1 - J(x)(1 + \lambda J(x)))\} \end{aligned} \quad (3.24)$$

where $R(t)$ is the CDF of the random variable T with pdf $r(t)$.

The PDF corresponding to the T-Transmuted X Family of Distributions in equation(3.24) is given by

$$g(x; \lambda) = G'(x) = \frac{j(x)[1 + \lambda - 2\lambda J(x)]}{1 - J(x)(1 + \lambda J(x))} r\{-\ln(1 - J(x)(1 + \lambda J(x)))\} \quad (3.25)$$

Some properties on the T-Transmuted X Family of Distributions:

- When $\lambda = 0$ the T-Transmuted X Family of Distributions reduces to T-X Family of Distributions with $W[F(x)] = -\log(1 - J(x))$.
- The HRF of the T-Transmuted X family is given by,

$$(x) = \frac{j(x)[1 + \lambda - 2\lambda J(x)]}{1 - J(x)(1 + \lambda J(x))} \frac{r\{-\ln(1 - J(x)(1 + \lambda J(x)))\}}{1 - R\{-\ln(1 - J(x)(1 + \lambda J(x)))\}}$$

Some examples of T-Transmuted X Family of distributions with different choices for the random variables T and X are Exponential-Transmuted Weibull (ETW), Exponential-Transmuted Rayleigh (ETR) distribution, Exponential-Transmuted Frechet (ETF) distribution and Lomax-Transmuted X Family of distributions.

3.4.1 EXPONENTIAL-TRANSMUTED EXPONENTIAL(ETE) DISTRIBUTION

In this section, we define and study some properties of a member of the T-Transmuted X family, the Exponential-Transmuted Exponential distribution. Jayakumar and Girish[25] applied T-X method and introduced (ETE)distribution.

Then $g(x; \lambda)$ in equation(3.25) reduces to

$$g(x; \theta, \beta, \lambda) = \theta \beta e^{-\theta \beta x} \frac{(1 - \lambda + 2\lambda e^{-\beta x})}{(1 - \lambda + \lambda e^{-\beta x})^{1-\theta}}; x > 0, \theta, \beta > 0, \lambda \in [-1, 1] \quad (3.26)$$

When $\lambda = 0$, ETE distribution becomes exponential distribution.

Figure 4 gives the graph of PDF of ETE distribution for various values of λ , β and θ

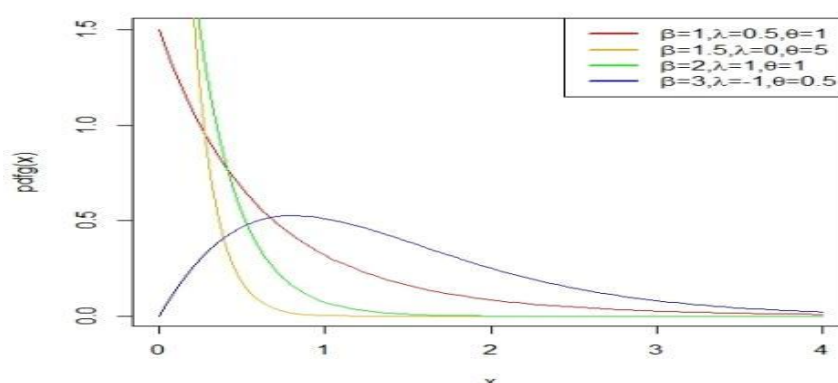


Figure 4: PDF of ETE for Various Values of λ , β and θ

The CDF of the ETE distribution is

$$G(x) = 1 - e^{-\theta \beta x} (1 - \lambda + \lambda e^{-\beta x})^\theta \quad (3.27)$$

The HRF of ETE distribution is

$$h(x) = \theta \beta \frac{(1 - \lambda + 2\lambda e^{-\beta x})}{(1 - \lambda + \lambda e^{-\beta x})}$$

The HRF is increasing if $-1 < \lambda < 0$ and decreasing if $0 < \lambda < 1$. When $\lambda = 0$ and $\lambda = 1$ the ETE distribution has a constant hazard rate.

Jayakumar and Girish[25]discovered the mathematical properties of this family such as ordinary moments, quantile and generating functions, modes and entropy. Also discussed the estimation of the parameters using maximum likelihood method and used simulation study to check the performance of the maximum likelihood estimate. The distribution is applied to two real data sets were fitted for the ETE and compared with four known distributions. The results showed that the ETE is a relatively better model to fit data than the other distributions.

In literature Clement and Girish[46] studied the further developments of T-Transmuted X family of distribution. Chhetri et al.[47] proposed and studied the Kumaraswamy Transmuted Pareto distribution. Also Afify et al.[48] studied the Beta Transmuted-H distribution. Similarly Chhetri et al.[49] introduced the Beta Transmuted Pareto distribution.

3.5 THE KUMARASWAMY MARSHALL-OLKIN FAMILY OF DISTRIBUTIONS

The Marshall-Olkin family of distributions was proposed by Marshall and Olkin[50] with CDF

$$F(x) = \frac{J(x)}{1 - cJ(x)}$$

for $c > 0$, where $J(x)$ is the CDF of the baseline distribution.

Alizadeh et al.[10] used $W[F(x)]$ to be the CDF Marshall-Olkin family of distributions and T to be the Kumaraswamy distribution in T - X family and declined the Kumaraswamy Marshall-Olkin family of distributions for generating distributions, the corresponding CDF is

$$G(x) = \int_0^{\frac{J(x)}{1-cJ(x)}} r(t)dt = 1 - \{1 - \{\frac{J(x)}{1-cJ(x)}\}^\alpha\}^\beta \quad (3.28)$$

where $r(t)$ is the PDF of the Kumaraswamy distribution.

The PDF corresponding to the Kumaraswamy Marshall-Olkin family of distributions in equation(3.28) is given by

$$g(x; \alpha, \beta, c) = G'(x) = \frac{\alpha\beta c j(x) J^{\alpha-1}(x)}{[1-cJ(x)]^{\alpha+1}} \{1 - \{\frac{J(x)}{1-cJ(x)}\}^\alpha\}^{\beta-1}; \alpha, \beta, c > 0 \quad (3.29)$$

Some properties on the Kumaraswamy Marshall-Olkin family of distributions:

- (a) When $c = 1$ the Kumaraswamy Marshall-Olkin family of distributions reduces to T - X

Family of Distributions with $W[F(x)] = J(x)$.

- (b) The HRF of the Kumaraswamy Marshall-Olkin family of distributions is given by

$$x(x) = \frac{\alpha\beta c j(x) J^{\alpha-1}(x)}{[1-cJ(x)]} \{[1 - cJ(x)]^\alpha - J^\alpha(x)\}$$

Some examples of Kumaraswamy Marshall-Olkin family of distributions with different choices for the random variable X are Kumaraswamy Marshall-Olkin Exponential (KwMO-E), Kumaraswamy Marshall-Olkin Lomax (KwMO-L), Kumaraswamy Marshall-Olkin Frechet (KwMO-Fr) and Kumaraswamy Marshall-Olkin Weibull (KwMO-W) Distribution.

3.5.1 THE KUMARASWAMY MARSHALL-OLKIN LOG-LOGISTIC DISTRIBUTION

In this section, we define and study some properties of a member of the Kumaraswamy Generalized Marshall family, the Kumaraswamy Marshall-Olkin Log-Logistic Distribution. Cakmakyapan et al.[51] applied T - X method and introduced Kumaraswamy Marshall-Olkin Log-Logistic Distribution.

Then $g(x; \alpha, \beta, c)$ in equation(3.29) reduces to

$$g(x; \alpha, \beta, c, \theta, \gamma) = \frac{\alpha\beta c \theta^\gamma x^{\gamma\alpha-1}}{[x^\gamma + c\theta^\gamma]} [1 - (\frac{x^\gamma}{x^\gamma + c\theta^\gamma})^\alpha]^\beta; x > 0, \theta, \beta > 0, \alpha, \gamma, c > 0 \quad (3.30)$$

When $c = 1$, The Kumaraswamy Marshall-Olkin Log-Logistic Distribution becomes the Kumaraswamy Log-Logistic Distribution.

Figure 5 gives the graph of PDF of The Kumaraswamy Marshall-Olkin Log-Logistic Distribution for various values of α, γ, β, c and θ

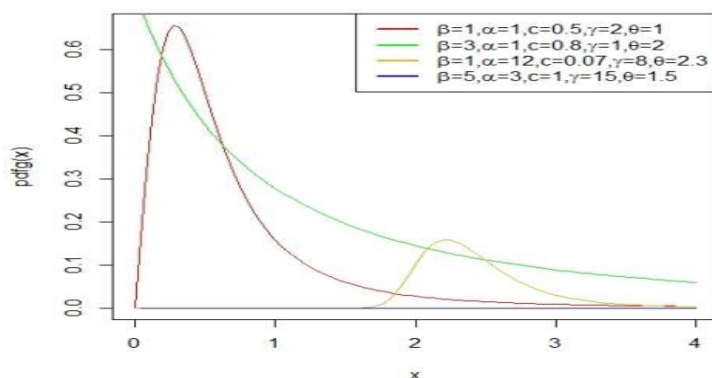


Figure 5: PDF of the Kumaraswamy Marshall-Olkin Log-Logistic Distribution For Various Values of α, γ, β, c and θ .

The CDF of the Kumaraswamy Marshall-Olkin Log-Logistic Distribution is

$$G(x) = 1 - \left[1 - \left(\frac{x^\gamma}{x^\gamma + c\theta^\gamma} \right)^\alpha \right]^\beta \quad (3.31)$$

The HRF of Kumaraswamy Marshall-Olkin Log-Logistic distribution is

$$h(x) = \frac{\alpha\beta c(\theta)^\gamma x^{\gamma\alpha-1}}{[x^\gamma + c(\theta)^\gamma]^\beta} \left(\frac{x^\gamma}{x^\gamma + c(\theta)^\gamma} \right)^{\alpha-1} \left[1 - \left(\frac{x^\gamma}{x^\gamma + c(\theta)^\gamma} \right)^\alpha \right]^{\beta-1}$$

The HRF has increasing, upside-down bathtub and bathtub shaped hazard rate function if $-1 < \lambda < 0$ and decreasing if $0 < \lambda < 1$.

Cakmakyapan et al.[51] discovered the mathematical properties of this family such as ordinary moments, quantile, generating functions, order statistics modes and Reliability. Also discussed the estimation of the parameters using maximum likelihood method. The distribution is applied to a real data sets were fitted for the Kumaraswamy Marshall-Olkin Log-Logistic distribution and compared with three known distributions. The results showed that the Kumaraswamy Marshall-Olkin Log-Logistic distribution is a relatively better model to fit data than the other distributions.

In literature Handique and Chakraborty[52] introduced Beta generated Kumaraswamy Marshall-olkin-G family of distributions, Alizadeh et al.[9]proposed the Beta Marshall-Olkin family of distributions. Similarly Abid and Hassan[53] introduced the Beta Marshall-Olkin Extended Uniform distribution.

4. CONCLUSION

This paper provides an overview of T-X family of distributions, except those discussed in Tomy and Jose[2]. These distributions provide flexible family in which new distributions can be defined and various existing distributions are special cases. Future research in this area is to generate more family of distributions, both continuous and discrete, and to develop their inferential procedures. For researchers and practitioners, we aspire, this review will give a summary of T-X method and associated references for further study in the theory and applications of statistical distributions.

REFERENCES

- [1] Alzaatreh, A.; Lee, C.; Famoye, F. "A New Method for Generating Families of Continuous Distributions", Metron (2013) , Vol.71, Issue.1, pp. 63-79.
- [2] Tomy, L.; Jose, M. "A Review Study on T-X Family of Distributions", In the proceedings of International Conference on statistics for Twenty-First Century (ICSTC 2018), Trivandrum, India, 2018. Submitted.
- [3] Eugene, N.; Lee, C.; Famoye, F. "Beta-normal distribution and its applications", Communications in Statistics-Theory and Methods (2002), Vol.31, Issue.4, pp.497-512.

- [4] Cordeiro, G.M. ; de Castro, M. "A new family of generalized distributions, Journal of Statistical Computation and Simulation (2011), Vol.**81**, Issue.**7**, pp.**883-898**.
- [5] Pescim, R.R. ; Cordeiro, G.M. ; Demetrio, C.G.B. ; Ortega, E.M.M; Nadarajah, S. "The new class of Kummer beta generalized distributions", Statistics and Operations Research Transactions (2012), Vol.**30** ,pp.**153-180**.
- [6] Alexander, C.; Cordeiro, G.M. ; Ortega, E.M.M.; Sarabia, J.M. "Generalized beta-generated distributions, Computational statistics and data analysis (2012), Vol.**56**, Issue.**6**, pp.**1880-1896**.
- [7] Tahir, M.H; Zubair, M. ; Cordeiro, G.M. ; Alzaatreh, A.; Mansoor, M. "The Poisson-X family of distributions", Journal of Statistical Computation and Simulation (2016), Vol.**86**, Issue.**14**, pp. **2901-2921**.
- [8] Akinsete, A.; Famoye, F.; Lee, C. "The Kumaraswamy-geometric distribution", Journal of Statistical Distributions and Applications(2014), Vol.**1**, Issue.**17**, pp. **1-21**.
- [9] Alizadeh, M.; Cordeiro, G.M. ; de Brito, E. ; Demetrio, C.G.B. "The beta marshall-olkin family of distributions", Journal of Statistical Distributions and Applications (2015), Vol.**2**, Issue.**4**, pp. **1-18**.
- [10] Alizadeh, M.; Tahir, M.H.; Cordeiro, G.M; Mansoor, M.; Zubair, M.; Hamedani, G.G. "The kumaraswamymarshall-olkin family of distributions", Journal of the Egyptian Mathematical Society(2015), Vol.**23**, pp. **546-557**.
- [11] Ristic, M.M.; Balakrishnan, N. "The Gamma Exponentiated Exponential Distribution, Journal of Statistical Computation and Simulation (2012), Vol.**82**, pp. **1191-1206**.
- [12] Zografos, K.; Balakrishnan, N. "On families of beta- and generalized gamma-generated distributions and associated inference, Statistical Methodology (2009), Vol.**6**, pp. **344-362**.
- [13] Alzaatreh, A.; Famoye, F; Lee, C. "Weibull-Pareto distribution and its applications", Communications in Statistics-Theory and Methods (2013), Vol.**42**, pp. **1-19**.
- [14] Cordeiro, G.M ; Ortega, E.M.; Popovic, B.V. ; Pescim, R.R. "The Lomax generator of distributions: Properties, minification process and regression model", Applied Mathematics and Computation (2014), Vol.**247**, pp. **65-86**.
- [15] Cordeiro, G.M.; Alizadeh, M.; Ortega, E.M. "The exponentiated half-logistic family of distributions: Properties and applications", Journal of Probability and Statistics (2014), Vol.**2014**, Issue.**1**, pp. **1-21**.
- [16] Jamal, F.; Nasir, M.A. "Generalized Burr X family of distribution", International Journal of Mathematics and Statistics (2018), Vol.**19**, Issue.**1**, pp. **55-73**.
- [17] Alzaghal, A.; Ghosh, I.; Alzaatreh, A. "On Shifted Weibull-Pareto Distribution", International Journal of Statistics and Probability (2016), Vol.**5**, Issue.**4**, pp. **139-149**.
- [18] Cordeiro, G.M.; Alizadeh, M.; Nascimento, A.D.C.; Rasekhi, M. "The ExponentiatedGompertz Generated Family of Distributions: Properties and Applications", Chilean Journal of Statistics (2016), Vol.**7**, Issue.**2**, pp. **29-50**.
- [19] Alzaatreh, A.; Famoye, F. ; Lee, C. "Gamma-Pareto distribution and its applications", Journal of Modern Applied Statistical Methods (2012), Vol.**11**, Issue.**1**, pp. **78-94**.
- [20] Alzaatreh, A. ; Lee, C.; Famoye,F. "On the discrete analogues of continuous distributions", Stat Methodology (2012), Vol.**9**, pp. **589-603**.
- [21] Tahir, M.H.; Zubair, M.; Cordeiro, G.M. ; Alzaatreh, A.; Mansoor, M. "The Weibull-Power Cauchy distribution: model, properties and applications" Hacettepe Journal of Mathematics and Statistics (2017), Vol.**46**, Issue.**4**, pp. **767-789**.
- [22] Alzaghal, A.; Famoye, F.; Lee, C. "Exponentiated T-X Family of Distributions with Some Applications", International Journal of Statistics and Probability (2013), Vol.**2**, Issue.**3**, pp. **31-49**.
- [23] Torabi, H.; Hedesh, N.M . "The gamma-uniform distribution and its applications",

Kybernetika (2012), Vol.**48**, Issue.**1**, pp. **16-30**.

[24] Tahir, M.H.; Cordeiro, G.M.; Alzaatreh, A.; Mansoor, M.; Zubair, M. "A New Weibull-Pareto Distribution: Properties and Applications", Communications in Statistics - Simulation and Computation (2016), Vol.**45**, Issue.**10**, pp. **3548-3567**.

[25] Jayakumar, K.; Girish, B.M. "T-transmuted X family of Distributions", Statistica (2017), Vol.**77**, Issue.**3**, pp. **251-276**.

[26] Merovci, F.; Elbatal, I. "Weibullrayleigh distribution: theory and applications", Applied Mathematics and Information Science (2015), Vol.**9**, Issue., pp. **2127-2137**.

[27] Yousof, H.M.; Afify, A.Z.; Hamedani, G.G.; Aryal, G. "The Burr X generator of distributions for lifetime data". Journal of Statistical Theory and Applications (2016), Vol.**16**, Issue.**3**, pp. **288-305**.

[28] Bourguignon, M.; Silva, R.B.; Cordeiro, G.M. "The Weibull-G family of probability distributions", Journal of Data Science (2014), Vol.**21**, Issue.**1**, pp. **53-68**.

[29] Cordeiro, G.M.; Alizadeh, M.; Tahir, M.H.; Mansoor, M.; Bourguignon, M.; Hamedani, G.G. "The beta odd log logistic generalized family of distributions", Hacettepe University Bulletin of Natural Sciences and Engineering Series B: Mathematics and Statistics (2015), Vol.**45**, Issue.**73**, pp. **1-19**.

[30] Oguntunde, P.E.; Balogun, O.S.; Okagbue, H.I.; Bishop, S.A. "The Weibull-Exponential Distribution: Its Properties and Applications", Journal of Applied Sciences (2015), Vol.**15**, pp. **1305-1311**.

[31] Chandrakant; Rastogi, M.K.; Tripathi, Y.M. "On a Weibull-Inverse Exponential Distribution", Annals of Data Science (2017), Vol.**5**, Issue.**2**, pp. **209-234**.

[32] Hosseini, B.; Afshari, M.; Alizadeh, M. "The Generalized Odd Gamma-G Family of Distributions: Properties and Applications", Austrian Journal of Statistics (2017), Vol.**10**, Issue.**1**.

[33] Torabi, H.; Montazeri, N.H. "The logistic-uniform distribution and its application", Communications in Statistics-Simulation and Computation (2014), Vol.**43**, pp. **2551-2569**.

[34] Al-Aqtash, R.; Lee, C.; Famoye, F. "Gumbel-Weibull distribution: Properties and applications" Journal of Modern Applied Statistical Methods (2014), Vol.**13**, pp. **201-225**.

[35] Tahir, M.H.; Cordeiro, G.M.; Alzaatreh, A.; Mansoor, M.; Zubair, M. "The Logistic-X Family of Distributions and Its Applications, Communication in Statistics- Theory and Methods (2014), Vol.**45**, Issue.**24**, pp. **7326-7349**.

[36] Aljarrah, M.A.; Lee, C.; Famoye, F. "On generating T-X family of distributions using quantile functions", Journal of Statistical Distributions and Applications (2014), Vol.**1**, Issue.**2**, pp. **1-17**.

[37] Alzaatreh, A.; Lee, C.; Famoye, F. "T-Normal family of distributions: a new approach to generalize the normal distribution", J. Stat. Distrib. Appl (2014)., Vol.**1**, pp. **1-16**.

[38] Nasir, M.A.; Tahir, M.H.; Alijarrah, M.; Jamal, F. "A New Generalized Burr Family of Distributions Based on Quantile Function, Journal of Statistics Applications and Probability (2017), Vol.**16**, Issue.**3**, pp. **499-504**.

[39] Alzaatreh, A.; Lee, C.; Famoye, F. "Family of generalized gamma distributions: properties and applications", Hacettepe Journal of Mathematics and Statistics, Vol.**5**, Issue.**3**, pp. **869-886**.

[40] Almheidat, M.; Famoye, F.; Lee, C. "Some Generalized Families of Weibull Distribution: Properties and Application", International Journal of Statistics and Probability, (2015), Vol.**4**, Issue.**3**, pp. **18-35**.

[41] Alzaatreh, A.; Lee, C.; Famoye, F.; Ghosh, I. "generalized Cauchy family of distributions with applications", J. Stat. Distrib. Appl. (2016), Vol.**3**, Issue.**12**, pp. **1-16**.

[42] Rooks, B.; Schumacher, A.; Cooray, K. "The power Cauchy distribution: derivation,

description, and composite models", NSF-REU Program Reports, (2010).

[43] Jain, G.C.; Consul, P.C. "A generalized negative binomial distribution", SIAM Journal on Applied Mathematics (1971), Vol.21, pp. 501-513.

[44] Hamedani, G.G. "Characterizations of Kumaraswamy-geometric distribution", Journal of Statistical Distributions and Applications, (2016), Vol.3, Issue.1.

[45] Shaw, W.; Buckley, I. "The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map", Research report. (2007).

[46] Clement, B.M.; Girish, B. "The further developments of T-Transmuted X family of distributions", Biostatistics and Biometrics (2018), Vol.7, Issue.4.

[47] Chhetri, S.; Long, H.; Aryal, G. "The Kumaraswamy transmuted Pareto distribution: Theory and applications", Journal of Statistical Distributions and Applications (2017), Vol.4, Issue.11, pp. 1-24.

[48] Afify, A.A.; Cordeiro, G.M.; Yousof, H.M.; Alzaatreh, A.; Nofal, Z.M. "The kumaraswamy transmuted-g family of distributions: properties and applications", Journal of Data Science (2016), Vol.14, pp. 245-270.

[49] Chhetri, S.; Akinsete, A.A.; Long, H.; Aryal, G. "The beta transmuted Pareto distribution: Theory and applications", Journal of Statistics Applications and properties (2017), Vol.6, Issue.2, pp. 243-258.

[50] Marshall, A.W.; Olkin, I. "A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families", Biometrika (1997), Vol.84, pp. 641-652.

[51] Cakmakyapan, S.; Ozel, G.; Gebaly, Y.M.H.E.; Hamedani, G.G. "The Kumaraswamy Marshall-Olkin Log-Logistic distribution", Journal of Statistical Theory and Applications (2017), Vol.17, Issue.1, pp.59-76.

[52] Handique, L.; Chakraborty, S. "A new beta generated Kumaraswamy Marshall Olkin-G family of distributions with Applications", Malaysia Journal of Science (2017), Vol.36, Issue.3, pp.157-174.

[53] Abid, S.H.; Hassan, H.A. "Beta Marshall-Olkin Extended Uniform distribution", Journal of Safety Engineering (2015), Vol.4, Issue.1, pp. 1-7.