

On the Spectra of Power Graphs of a Special Class of Finite Groups

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Abstract Let G be a finite group. The power graph $\text{pg}(G)$ is an undirected graph with vertex set G where two distinct vertices u and v in $\text{pg}(G)$ are adjacent if $u = v^\alpha$ or $v = u^\beta$ for some α, β in $\mathbb{N}$. We compute the spectrum of the power graphs of various finite groups and study their respective spectral properties. We also study the Laplacian spectrum and the corresponding spectral properties for these graphs.

Keywords : Power graph, Laplacian Spectrum, Classification : Finite Abelian groups, Finite non-abelian groups.

1 Introduction

According to literature, several graphs have been constructed from various algebraic structures. One such graph is the *power graph*. Chakrabarty et.al. [1] has defined the power graph for a semigroup S as a graph with vertex set S where two distinct vertices u and v in $\text{pg}(G)$ are adjacent if $u = v^\alpha$ or $v = u^\beta$ for some α, β in $\mathbb{N}$.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. Then the union [4] of Γ_1 and Γ_2 , denoted by $\Gamma_1 \cup \Gamma_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The join of Γ_1 and Γ_2 , denoted by $\Gamma_1 \vee \Gamma_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ together with all the edges joining all the vertices in V_1 to all the vertices in V_2 . For any connected graph Γ , $n\Gamma$ represents a graph with n components each isomorphic to Γ .

Isomorphism between two graphs (groups) G_1 and G_2 is denoted by $G_1 \cong G_2$. For any positive integer n , $\varphi(n)$ represents the Euler-Totient function. For any group G , $\text{Aut}(G)$ denotes the group of automorphisms on G .

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an undirected graph. The Laplacian matrix corresponding to Γ denoted by $L(\Gamma) = [l_{ij}]$ is defined by

$$l_{ij} = \begin{cases} d(v_i) & \text{if } v_i = v_j \\ -1 & \text{if } v_i \neq v_j \text{ and } (v_i, v_j) \text{ forms an edge} \\ 0 & \text{if } v_i \neq v_j \text{ and } (v_i, v_j) \text{ does not form an edge} \end{cases}$$

The matrix $L(\Gamma)$ is symmetric and positive semidefinite and consequently, the eigenvalues are all non-negative real numbers. Since the sum of each row(column) is 0, it is an eigenvalue of $L(\Gamma)$. The eigenvalues of $L(\Gamma)$ are called Laplacian eigenvalues. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are Laplacian eigenvalues of Γ with multiplicities $m(\lambda_1), m(\lambda_2), \dots, m(\lambda_n)$, then

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_n) \end{pmatrix}$$

constitute the Laplacian spectrum of Γ denoted by $L.Spec(\Gamma)$.

Let Γ be a graph of order n . Then $\det(xI_n - L(\Gamma))$ is termed the Laplacian characteristic polynomial of Γ and is denoted by $\theta(\Gamma, x)$. For the null graph Γ , $\theta(\Gamma, x)$ is defined as 1.

In our work, we have tried to characterise the power graphs of some well-known abelian groups as well as non-abelian groups and also tried to determine the Laplacian spectrum of the corresponding power graphs so as to derive the spectral properties of these graphs.

2 Materials and Methods

In this section, we discuss some important results which are necessary for further reading.

Theorem 2.1 [1] Let G be a finite group. Then $g(G)$ is complete if and only if G is a cyclic group of order one or p^α for some prime p and $\alpha \in \mathbb{N}$.

Theorem 2.2 [3] For any graph Γ , the multiplicity of 0 as an eigenvalue of $L(\Gamma)$ is equal to the number of components of Γ .

Theorem 2.3 [2] If Γ is the disjoint union of graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_r$, then

$$\theta(\Gamma, x) = \prod_{i=1}^r \theta(\Gamma_i, x)$$

Theorem 2.4 [2] If Γ_1 and Γ_2 are disjoint graphs on n_1 and n_2 vertices respectively, then

$$\theta(\Gamma_1 \vee \Gamma_2, x) = \frac{x(x - n_1 - n_2)}{(x - n_1)(x - n_2)} \theta(\Gamma_1, x - n_2) \theta(\Gamma_2, x - n_1)$$

In addition to these pre-requisites, GeoGebra 5.0.562.0 was used to plot the power graph of the non-abelian group S_4 and its characterisation was further verified.

3 Results

We have characterised the power graphs of some well-known finite groups and computed their respective Laplacian spectrum. To keep things precise, we have categorised them as – a) Finite abelian groups b) Finite non-abelian groups

a) Finite abelian groups

In this section, we characterise the power graphs of certain finite abelian groups in terms of various combinations of the complete graph K_n using the various graph operations. We also compute the corresponding Laplacian Spectrum for each of these power graphs.

Theorem 3.1 For any prime p , $g(\mathbb{Z}_p \oplus \mathbb{Z}_p) \cong (p+1)K_{p-1} \vee K_1$ and $L.Spec(g(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is given by $\begin{pmatrix} 0 & 1 & p & p^2 \\ 1 & p-2 & (p-1)(p-2) & 1 \end{pmatrix}$

Proof. For any prime p , $\mathbb{Z}_p \oplus \mathbb{Z}_p$ contains $p^2 - 1$ elements of order p . Thus we have $\frac{p^2-1}{\phi(p)} = p+1$ cyclic subgroups of order p . Observe that these subgroups have only identity in common. Thus we have $p+1$ copies of K_p sharing a common identity, which can be viewed as $p+1$ copies of K_{p-1} . Since identity is adjacent to every vertex, we have, $g(\mathbb{Z}_p \oplus \mathbb{Z}_p) \cong (p+1)K_{p-1} \vee K_1$. Applying theorem 2.4, the latter result follows.

Corollary 3.1 $g(Klein\ 4)$ is isomorphic to the star graph $K_{1,3}$

Proof. Applying theorem 3.1, $g(Klein\ 4) \cong 3K_1 \cup K_1 \cong K_{1,3}$

Theorem 3.2 For any prime p and $n \in \mathbb{N}$,

$$g(\mathbb{Z}_{p^n}) \cong K_{p^{n-1}-1} \vee K_{\phi(p^n)+1} \text{ and } L.Spec(g(\mathbb{Z}_{p^n})) = \begin{pmatrix} 0 & p^n \\ 1 & p^n-1 \end{pmatrix}$$

Proof. Observe that every element of order p^k is adjacent to elements of order $p, p^2, \dots, p^{k-1}$ for $2 \leq k \leq n$. Mimic the proof in theorem 3.1 to obtain

$$g(\mathbb{Z}_{p^n}) \cong K_{\phi(p)} \vee K_{\phi(p^2)} \vee \dots \vee K_{\phi(p^{n-1})} \vee K_{\phi(p^n)+1}$$

$$\cong K_{p^{n-1}-1} \vee K_{\varphi(p^n)+1}$$

Applying theorem 2.1 or theorem 2.4, we get $L.Spec(g(\mathbb{Z}_{p^n})) = \begin{pmatrix} 0 & p^n \\ 1 & p^n-1 \end{pmatrix}$

Theorem 3.3 If p and q are two distinct primes, then

$$g(\mathbb{Z}_{p^2q}) \cong \{[K_{\varphi(pq)} \vee (K_{p-1} \cup K_{q-1})] \cup K_{\varphi(p^2)}\} \vee K_{\varphi(p^2q)+1}$$

Proof. Observe that every element of order pq is adjacent to elements of order p and q . Thus every vertex in $K_{\varphi(pq)}$ is adjacent to vertices in K_{p-1} and in K_{q-1} . But an element of order pq is not adjacent to an element of order p^2 . Thus $g(\mathbb{Z}_{p^2q})$ contains a copy of $[K_{\varphi(pq)} \vee (K_{p-1} \cup K_{q-1})] \cup K_{\varphi(p^2)}$. Also note that every element of order p^2q and identity is adjacent to all other vertices. Hence the result follows.

b) Finite Non-abelian groups

In this section, we determine the power graphs of some finite non-abelian groups in terms of the complete graph K_n using graph operations.

Theorem 3.4 $g(S_3) \cong K_1 \vee (3K_1 \cup K_2)$

Proof. S_3 contains a subgroup isomorphic to $\mathbb{Z}_3$ and three copies of subgroups isomorphic to $\mathbb{Z}_2$. By theorem 2.1, $g(\mathbb{Z}_3)$ is isomorphic to K_3 and $g(\mathbb{Z}_2)$ is isomorphic to K_2 . Thus $g(S_3)$ contains a copy of K_3 and three copies of K_2 sharing a common identity. Thus, $g(S_3) \cong K_1 \vee (3K_1 \cup K_2)$

Theorem 3.5 $g(Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2))$ and $g(S_3)$ are cospectral

Proof. $Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \{\varphi_e, \varphi_a, \varphi_b, \varphi_c, \varphi_1, \varphi_2\}$ where φ_a represents the automorphism map which fixes the element a and interchanges the other two non-trivial elements. φ_1 maps a to b , b to c and a to c . Similarly φ_2 sends a to c , b to a , c to b . Thus $Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong S_3$. Thus, their respective power graphs are also isomorphic and so they are cospectral.

Theorem 3.6 $g(Aut(S_3))$ and $g(S_3)$ are cospectral

Proof. Proof follows from the fact that $Aut(S_3) \cong S_3$

Theorem 3.7 $g(A_4) \cong K_1 \vee [4K_2 \cup 3K_1]$

Proof. A_4 contains three elements of order 2 and eight elements of order 3. Thus, there are three subgroups isomorphic to $\mathbb{Z}_2$ and four subgroups isomorphic to $\mathbb{Z}_3$. By theorem 2.1, $g(\mathbb{Z}_3)$ is isomorphic to K_3 and $g(\mathbb{Z}_2)$ is

isomorphic to K_2 . Thus, $g(A_4)$ contains three copies of K_2 and four copies of K_3 sharing a common identity. Thus, $g(A_4) \cong K_1 \vee [4K_2 \cup 3K_3]$

Theorem 3.8 $g(S_4) \cong K_1 \vee [6K_1 \cup 4K_2 \cup 3K_3]$

Proof. S_4 contains six subgroups isomorphic to $\mathbb{Z}_2$, four subgroups isomorphic to $\mathbb{Z}_3$ and three subgroups isomorphic to $\mathbb{Z}_4$. By theorem 2.1, $g(\mathbb{Z}_3)$, $g(\mathbb{Z}_2)$ and $g(\mathbb{Z}_4)$ are isomorphic to K_3 , K_2 and K_4 respectively. Thus, $g(S_4)$ contains six copies of K_2 , four copies of K_3 and three copies of K_4 sharing a common identity. Thus, $g(S_4) \cong K_1 \vee [6K_1 \cup 4K_2 \cup 3K_3]$.

Theorem 3.9 $g(\text{Aut}(S_4))$ and $g(S_4)$ are cospectral

Proof. Proof follows from the fact that $\text{Aut}(S_4) \cong S_4$

Theorem 3.10 $g(\text{Aut}(A_4))$ and $g(S_4)$ are cospectral

Proof. Proof follows from the fact that $\text{Aut}(A_4) \cong S_4$

4 Discussion

Chakrabarty et.al. [1] has characterized the power graph of a cyclic group of prime power order as seen in theorem 2.1

Following the footprints of the already established result, we have tried to characterize the power graphs of cyclic groups of order pq , p^2q , p^2q^2 . During the literature survey, I found that much work has been carried out on characterising the power graphs of cyclic groups only. So we started to think of characterising the power graphs of finite non cyclic groups such as $\mathbb{Z}_p \oplus \mathbb{Z}_p$, $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, $\mathbb{Z}_p \oplus \mathbb{Z}_{pq}$ and so on. Also, we observed that much study has already been done in characterization of power graphs of non-cyclic groups such as quaternion groups Q_n , dihedral groups D_{2n} of order $2n$. Since scope of my work lies on studying the spectral properties of graphs (as the title suggests), we have tried to bridge the gap between the work which we carried out so far and the spectral properties (like algebraic connectivity, spectral radius etc.) of those graphs.

So, we have compartmentalized the work into two steps-

Step 1. Characterizing the power graphs of certain finite groups (whose study has not been carried out yet) and

Step 2. Studying the spectral properties of the graphs.

In the due course we have further divided the work into two segments, that is, investigating the case of (a) Finite Abelian Groups (b) Finite Non-Abelian Groups.

To put it simple, we have applied both the steps to abelian and non-abelian groups separately.

5 Conclusion

Mohar [2] in his paper has characterised the power graph of the cyclic group $\mathbb{Z}_p^n$ but has not formulated it. But in this paper, we have attempted to formulate it in terms of the well-known complete graph K_n . Likewise, he has characterised and formulated the power graph of the abelian group $\mathbb{Z}_{pq}$. Motivated by his work, we have characterised and formulated the power graph of the abelian group $\mathbb{Z}_{p^2q}$. Although Mohar has determined the spectrum of power graph of $\mathbb{Z}_p \oplus \mathbb{Z}_p$, he has not formulated the power graph. A work in this regard was successfully carried out. A similar characterisation of other finite abelian groups in this regard is currently in progress. In the latter section, we have characterised the power graphs of some finite non-abelian groups. A generalisation of the work in this regard is currently in progress. Not to mention the limitations, the work has its own flaws. In most of the cases, a generalisation is very much necessary and in the latter section, we have only formulated the power graphs but the determination of the Laplacian spectrum of the corresponding power graphs is intentionally left behind as it is too obvious. But when a generalised result comes into play, we intend to determine the entire Laplacian Spectrum and its spectral properties.

References

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