THINK INDIA JOURNAL

ISSN:0971-1260
Vol-22- Issue-14-December-2019

Applications Of Differential Equations In Population Growth

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ABSTRACT

We cannot have a sustainable planet without stabilizing population. As human population increases, humans demand for resources like water, land, trees, and energy also increases. Unfortunately, the price for all this "increase and demand" is paid by the other endangered plants, animals and natural resources in an increasingly volatile and dangerous climate. Charles Darwin's famous quote "survival of the fittest" which states that individuals will compete (with members of their own or other species) for limited resources. The successful ones are more likely to survive. This necessitates a mathematical model by using differential equations to predict the future population in terms of growth rate and population figures with reasonably virtuous accuracy. The present work deals with applications of differential equations in population growth using exponential and logistic growth models, with which we can study the changes in size of populations through time.

Keywords: Population growth, Logistic Growth model, Exponential Growth model, Growth rate, Differential equations, Mathematical modelling.

I. INTRODUCTION

Projection of any country's population plays a significant role in the planning as well as in the decision making for the socio-economic and demographic development. Today the major issue of the world is the tremendous growth of the population. Any truly meaningful conservation and sustainability efforts must take the expanding human population footprint into consideration. Globally, over several thousands of people are added every day — each needs sufficient land, water, shelter, food, and energy for a decent life. In order to provide better living condition to the people in terms of basic needs like food, water, education, health care etc., the society requires planning and executing appropriate schemes in time bound manner. Mathematical Modelling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences which is created in the hope that the behaviour it predicts will resemble the real behaviour on which it is based. It involves the following processes.

(1) The mimicking of a real-world problem in mathematical terms: thus the construction of mathematical model by using Differential equations.

(2) The analysis or solution of the resulting mathematical problem.

ISSN:0971-1260
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(3) The interpretation of the mathematical results in the context of the original situation.

A model can be in many shapes, sizes and styles. It is important to highlight that a model is not reality but merely a human construct (representing the reality as much as possible) to help us better understand real-world system. One uses models in order to extract the important trend from complex processes to permit comparison among systems to facilitate analysis of causes of processes acting on the system and to make a prediction about the future.

At first glance, modelling the growth of a species would seem to be impossible since the population of any species always changes by integer amounts. Hence, the population of any species can never be a differentiable function of time. If a given population is very large and it is suddenly increased by one unit, then the difference is very small when compared to the given population. Therefore, we make the approximation such that the large populations change continuously and even differentiable with time. The projection of future populations are normally based on present population and reliable growth rate and the first order differential equations govern the growth of various species. Further, in any species un controlled exponential growth is not viable as the population and the growth rate are limited by the various factors which determine the sustainability and the growth rate. Hence, in this paper, mathematical modelling for global population growth based on exponential and logistic growth models is presented. The use of these growth models are widely established in many fields of modelling and forecasting.

II. MATERIALS AND METHOD

A research is best understood as a process of arriving at dependent solutions to the problems through the systematic collection, analysis and interpretation of data. In this paper, secondary population of the world from 1960 to 2010 (inclusive) were collected from UN World Population Prospectus (2017 Revision). The exponential and logistic growth mathematical model was used to compute the predicted population values.

III. DEVELOPMENT OF THE MODEL

The Exponential Growth Model:

In 1798 Thomas R. Malthus proposed a mathematical model of population growth. He proposed by the assumption that the population grows at a rate proportional to the size of the population. This is a reasonable assumption for a population of a bacteria or animal under ideal conditions (unlimited environment, adequate nutrition, absence of predators, and immunity from disease).

Supposing we know the population P_0 at some given time $t=t_0$, and we are interested in protecting the population P, at some future time $t=t_1$, i.e. to find the population function P(t) for $t_0 \leq t \leq t1$ that satisfies $P(t_0)=P_0$.

Then considering the initial value problem

$$\frac{dP}{dt} = kP(t)\ t_0 \leq t \leq t_1; P(t_0) = P_0 \qquad (1)$$

Integrating by variable separable in (1)

$$\int \frac{dP}{P} = k\int dt$$

$$\ln P = kt + c$$

ISSN:0971-1260
Vol-22- Issue-14-December-2019

$P(t) = P_0 e^{kt}$

$P(t) = P_0 \exp\{k(t - t_0)\}$ (2)

Where k is a constant known as the Malthus factor, is the multiple that determines the growth rate. Equation (1) is the Exponential growth model with (2) as its solution. It is a differential equation because it contains an unknown function P and it derivative $\frac{dP}{dt}$. Having formulated the model, we now look at its consequences. If we rule out a population of 0, then P(t)>0 for all t. So if k>0 then equation shows that $\frac{dP}{dt} < 0$ for all t. This means that the population is always increasing. In fact, as P(t) increases, equation (1) shows that $\frac{dP}{dt}$ becomes larger. In other words, the growth rate increases as the population increases. Equation (1) is appropriate for modelling population growth under ideal conditions, thus we have to recognize that a more realistic must reflect the fact a given environment has a limited resources.

The Logistic Growth Model:

A Belgian Mathematician Verhulst, showed that the population growth not only depends on the population size but also on how far this size is from its upper limit i.e. its carrying capacity (maximum supportable population). He modified Malthus's Model to make the population size proportional to both the previous population and a new term

$\frac{a - bP(t)}{a}$ (3)

Where a and b are called the vital coefficients of the population. This term reflects how far the population is from its maximum limit. However, as the population value grows and gets closer to $\frac{a}{b}$, this new term will become smaller and get to zero, providing the right feedback to limit the population growth. Thus the second term models the competition for available resources, which tends to limit the population growth. So the modified equation using this new term is:

$\frac{d}{dt} P(t) = \frac{aP(t)(a - bP(t)}{a}$ (4)

This is a nonlinear deferential equation unlike equation (1) in the sense that one cannot simply multiply the previous population by a factor. In this case the population P(t) on the right of equation (4) is being multiplied by itself. This equation is known as the logistic law of population growth. Putting P = P_0 for t = 0, whereP_0 represents the population at some specified time, t = 0, equation (4) becomes

$$\frac{d}{dt} P = aP - bP^2 \qquad (5)$$

Separating the variables in equation (5) and integrating, we obtain

$$\int \frac{1}{a} \left(\frac{1}{P} + \frac{b}{a - bP}\right) dP = t + c$$

So that $\frac{1}{a}(\log P - \log(a - bP)) = t + c$ (6)

Using t=0 and P=P_0, we see that

$$c = \frac{1}{a}(\log P0 - \log(a - bP0))$$

Equation (6) becomes,

$$\frac{1}{a}(\log P - \log(a - bP)) = t + \frac{1}{a}\log P0 - \log(a - bP0)$$

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Solving for P gives

$$P = \frac{\frac{a}{b}}{1 + (\frac{\frac{a}{b}}{N_0} - 1)e^{-at}} \quad (7)$$

Taking the limit of equation (7) as t→∞, we get (science a >0)

$$P_{\max} = \lim_{t\to\infty} P = \frac{a}{b} \quad (8)$$

If the time t=1 and t=2, then the values of P are P1 and P2 respectively, from (8) we get,

$$\frac{b}{a}(1-e^{-a}) = \frac{1}{P1} - \frac{e^{-a}}{P_0}, \frac{b}{a}(1-e^{-2a}) = \frac{1}{P_2} - \frac{e^{-2a}}{P_0} \quad (9)$$

Dividing the members of the second equation in relation (9) by corresponding Members of the first equation to eliminate $\frac{b}{a}$ we get

$$1 + e^{-1} = \frac{\frac{1}{P_2} - \frac{e^{-2a}}{P_0}}{\frac{1}{P_1} - \frac{e^{-a}}{P_0}} \quad (10)$$

So that, $e^{-a} = \frac{P_0(P_2 - P_1)}{P_2(P_1 - P_0)}$ (11)

Substituting the values of e^{-a} in to the (9) equation, we obtain

$$\frac{b}{a} = \frac{P_1^2 - P_0P_2}{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)} \quad (12)$$

Thus limiting the value of P, we get

$$P_{\max} = \lim_{t\to\infty} P = \frac{a}{b} = \frac{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)}{P_1^2 - P_0P_2} \quad (13)$$

IV. MEAN ABSOLUTE PERCENTAGE ERROR

It is an evaluation statistic which is used to assess the goodness of fit of different models in national and sub national population projections. This statistic is expressed in percentage. The concept of mean absolute percentage error (MAPE) seems to be the very simple but of great importance in the selecting a parsimonious model than the other statistics. A model with smaller MAPE is preferred to the others models.

The mathematical form of MAPE is given under

$$\text{MAPE} = \frac{1}{P}\sum_{t=}^{P}\left|\frac{Y_t - Y}{Y_t}\right| \times 100$$

Where Y_t, Y and P are the actual, fitted and number of observation of the (dependent variable) population respectively.

V. RESULTS

For the estimation of the future population of the world, we need to determine the growth rate of the world using exponential growth model in (2). Using the actual population of the world in thousands with t=0, we have $P_{0=}3033213$, $P_1=3700578$ when t=1. We can solve for the growth rate k

$$3700578 = 3033213e^k$$

$$k = \ln(\frac{3700578}{3033213})$$

$$k = 0.178925985$$

Hence the general solution,

$$P(t) = 3033213e^{0.178925985k}$$

This means that the predicted growth rate of the world population is 17.8 % for ten

ISSN:0971-1260
Vol-22- Issue-14-December-2019

years. For the exponential growth model in (2) with this we projected the population of the world to 2050.
Based on table 1, let t=0,1and 2 correspond to the year 1960, 1961 and 1962 respectively.
Then and correspond to 3033213, 3700578 and 4458412.

Substituting the values of P0, P1, and P2 into equation (13) we get

$$P_{\max} = \frac{a}{b} = 14647743$$

This is the predicted carrying capacity or limiting value of the population of the world.

From equation (11), we obtain $e^{-a} =$ 0.772562.

So therefore a=-ln (0.772562)

Solving this for a, we get

a = 0.258043.

This implies that the predicted growth rate of the world's population is 25.8% for ten years.

From a/b =14647743

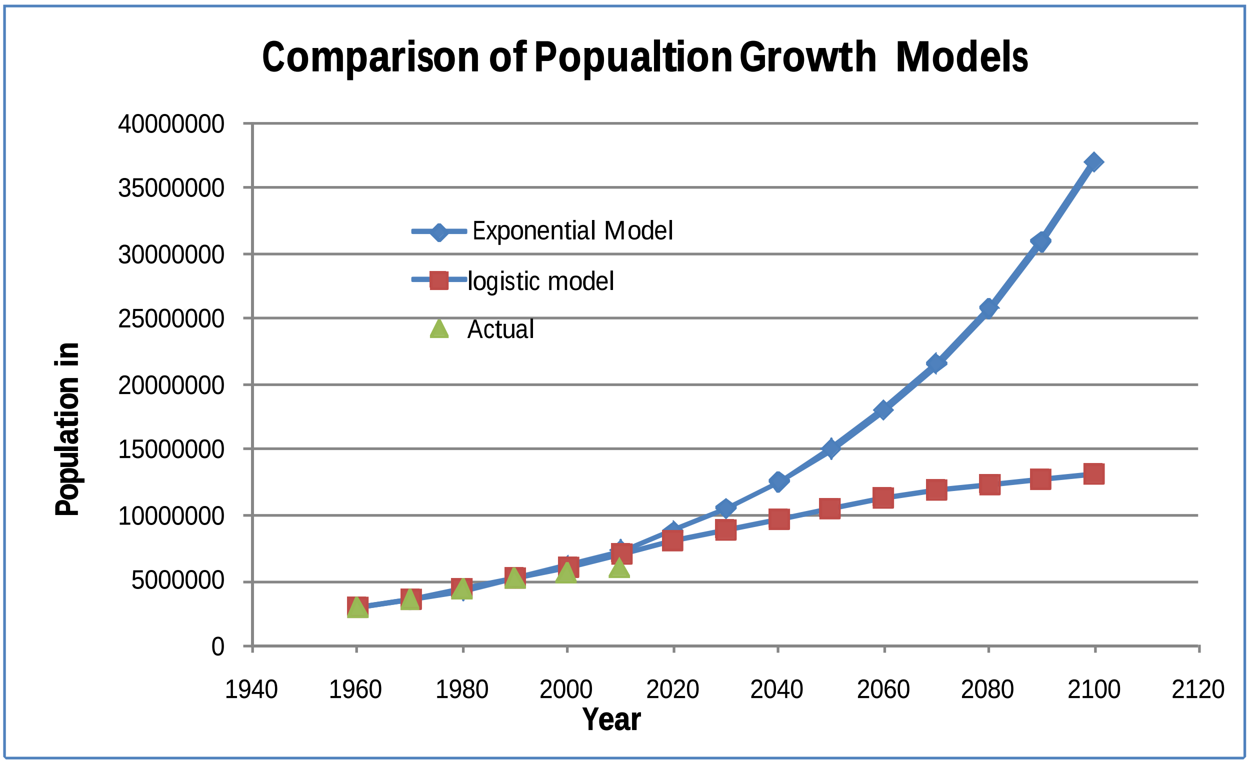
We have b=1.7*10^8

Substituting the values of P_0, e^a, $\frac{a}{b}$ in (7) gives

$$P = \frac{14647763}{1 + (3.829118)(0.772562)^t}$$

Years	Actual Population	Projected Population	
		Exponential Growth	Logistic Growth
1960	3033213	3033213	3033213
1970	3700578	3627517.167	3700578
1980	4458412	4338264.671	4458412
1990	5330943	5188270.514	5296356.151
2000	5751474	6204819.892	6196021.903
2010	6145007	7420544.049	7131957.891
2020		8874467.744	8074208.757
2030		10613261.94	8992008.315
2040		12692741.95	9857685.994
2050		15179659.1	10649773.69
2060		18153843.46	11354637.01
2070		21710766.38	11966516.13
2080		25964605.1	12486345.86
2090		31051907.91	12919943.96
2100		37135977.27	13276113.14
Mean Absolute Error Percentage		**5.997501819%**	**4.073183688%**

ISSN:0971-1260
Vol-22- Issue-14-December-2019



VI. CONCLUSION

In conclusion the Exponential Model predicted a growth rate of approximately 1.7% and predicted World's population to be 37135977.27 thousands in the year 2100 with a MAPE of 5.997501819%. The Logistic Model on the other hand predicted a carrying capacity for the population of the world to be 14647743. Here we found out that the vital coefficients a and b are 0.258043 and 1.7×10^{8} respectively. Thus the population growth rate of the world according to this model is approximately 2.5% per annum. It also predicted the population of the world to be13276113.14 thousand in 2100 with a MAPE of 4.073183688%. Hence we can conclude that the Logistic Model gave a good forecasting result as compared to the Exponential model.

The more industrialized a Nation is, the more living space and food it has and the smaller the coefficient b, thus raising the carrying capacity.

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